SLOW OSCILLATIONS OF POTENTIAL IN ELECTRON-ION CURRENTS CLOSE TO THE EMITTER

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This paper is concerned with quasi-stationary potential distributions in one-dimensional ion-electron currents close to an emitting surface. It is assumed that the part played by near collisions can be neglected. Three types of regimes unstable with respect to slow changes in the boundary conditions have been found. The instability is attributable to the feedback created by slow particles reflected to the emitter by potential barriers in the Debye layer close to the emitter.

Problems of the stability of stationary oscillations of potential in relation to rapid fluctuations occurring in a particle flux have been considered by a number of authors in the hydrodynamic [1–3] and kinetic [3–7] approximations. Reference [8] gives a classification of the stationary one-dimensional potential distributions in electron-ion fluxes emitted from a surface (on the assumption that the role of near collisions can be neglected).

Below, we shall investigate the instabilities in quasi-stationary ion-electron fluxes that develop due to feedback between the emitter field E_0 and the flow of particles to infinity j_0 at fixed initial ion and electron velocity distribution functions $f_{0i}(v)$ and $f_{0e}(v)$ and a fixed value of the current j_k supplied to the emitter by the power source. The feedback is attributable to slow particles reflected to the emitter by potential barriers.

Instabilities of the first and second types due to the absence of stationary regimes at adjacent values of E_0 develop during a time of the order of $1/\omega_0$ (ω_0 is the plasma frequency), since the number of reflected particles responsible for feedback is comparable with the total number of particles.

Instability of the third type is characteristic of those potential distributions when slow particles leaving the emitter are first accelerated and only then retarded and reflected to the emitter. In particular, instability is observed in the case of a homogeneous ion-electron flux with zero potential relative to the emitter in the presence of particles of the same sign with energies close to zero. The development time for an instability of the third type may substantially exceed $1/\omega_0$, if the number of accelerated slow particles is relatively small.

In §1 the form of the approximate system of equations is established for the case of slow oscillations of the boundary conditions in time, in §2 solutions of this system are investigated.

We shall introduce certain basic notation: x is the distance from the emitting surface, t is time, E is the electric field, φ is the electrostatic potential with reversed sign, m, -e are the electronic mass and charge, μ is the electron-ion mass ratio, v is the particle velocity in the direction of the x axis, v_{im}, v_{em} are the minimum initial velocities of ions and electrons, starting from which the particles escape to infinity, $-\varphi_{im}$, φ_{em} are the minimum and maximum values of the potential.

§1. Let the time scale τ, given by the variation of the boundary conditions in time, be much greater than $1/\omega_0$. For simplicity, we shall consider the case with fixed ions of constant density n_i.

We shall start from the system of equations

$$\frac{\partial f(x,v,t)}{\partial t} + v\frac{\partial f(x,v,t)}{\partial x} - \frac{e}{m}E(x,t)\frac{\partial f(x,v,t)}{\partial v} = 0,$$
$$\frac{\partial E(x,t)}{\partial x} = 4\pi e\left(n_i - \int_{-\infty}^{\infty} f(x,v,t)\,dv\right), \tag{1.1}$$

with boundary conditions

$$E(x,t) = E_0(t), \quad f(x,v,t) = f_0(v,t) \quad \text{at } x=0.$$

Here $f(x, v, t)$ is the electron distribution function. We introduce the dimensionless quantities

$$x' = \frac{x}{r_d}, \quad v' = \frac{v}{v^\circ}, \quad t' = \frac{t}{\tau}, \quad E' = \frac{E}{E^\circ},$$
$$f' = \frac{fv^\circ}{n_i}, \quad r_d \equiv \frac{v^\circ}{\omega_0}, \quad \omega_0^2 \equiv \frac{4\pi e^2 n_i}{m}, \quad E^\circ \equiv \frac{mv^{\circ 2}}{er_d}. \tag{1.2}$$

Here v° is the velocity scale. The quantities df/dt', $v'df'/dx'$, f' are of the same order. Substituting (1.2) into (1.1) and omitting the primes from the new variables, we obtain

$$\varepsilon\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - E\frac{\partial f}{\partial v} = 0,$$
$$\frac{\partial E}{\partial x} = 1 - \int_{-\infty}^{\infty} f\,dv \quad \left(\varepsilon = \frac{1}{\tau\omega_0}\right). \tag{1.3}$$

Here ε is a small parameter equal to $1.8 \cdot 10^{-5}\tau/n_i^{1/2}$, if τ is measured in seconds, and n_i in cm^{-3}.

Transferring the first term in the kinetic equation to the right side and formally integrating the equation as inhomogeneous along the characteristics, we obtain the kinetic equation in the form

$$f(x,v,t) = f_1(x,v,t) - \varepsilon\int_0^x \frac{dx'}{v'}\frac{\partial f}{\partial t}(x',v',t) \qquad \left(\varphi \equiv \int_0^x E\,dx\right)$$
$$f_1(x,v,t) \equiv f_0\left(\sqrt{v^2 + 2\varphi(x,t)}, t\right),$$
$$v' \equiv \sqrt{v^2 + 2\varphi(x,t) - 2\varphi(x',t)}. \tag{1.4}$$

Consequently, when $x\varepsilon \ll 1$ it is possible to represent the solution in series form:

$$f(x,v,t) =$$
$$= f_1(x,v,t) - \varepsilon\int_0^x \frac{dx'}{v'}\frac{\partial f_1}{\partial t}(x',v',t) + \varepsilon^2\int_0^x dx'\int_0^{x'} dx'' \ldots, \tag{1.5}$$

in powers of the parameter $x\varepsilon$ (or $x/\tau v^\circ$, if x is in cm, v° in cm/sec, and τ in sec).

From the equations of system (1.3) there follows the relation

$$\frac{E^2(x,t)}{2} = \varphi(x,t) +$$
$$+ \int_{-\infty}^{\infty} f(x,v,t)\,v^2\,dv + \varepsilon\int_0^x dx'\int_{-\infty}^{\infty}\frac{\partial f(x',v,t)}{\partial t}v\,dv + C(t) \tag{1.6}$$

where

$$C(t) \equiv \frac{E_0^2(t)}{2} - \int_{-\infty}^{\infty} f_0(v, t)\, v^2\, dv.$$

Substituting (1.5) in (1.6), we obtain

$$\frac{E^2(x, t)}{2} = \varphi(x, t) + \int_{-\infty}^{\infty} f_1(x, v, t)\, v^2\, dv +$$

$$+ \varepsilon \int_0^x dx' \int_{-\infty}^{\infty} \frac{\partial f_1(x', v, t)}{\partial t} \left(v - \sqrt{v^2 - 2\varphi(x, t) + 2\varphi(x', t)}\right) dv +$$

$$+ \varepsilon^2 \int_0^x dx' \int_0^{x'} dx'' \ldots + C(t). \qquad (1.7)$$

Thus, close to the emitting surface, when $x\varepsilon \ll 1$ (in this case x is reckoned in Debye radii), it is possible to use the relation

$$\frac{E^2(x, t)}{2} =$$

$$= \varphi(x, t) + \int_{-\infty}^{\infty} f_0(v, t)\, v \sqrt{v^2 - 2\varphi(x, t)}\, dv + C(t), \qquad (1.8)$$

which is the integral of the zero-th-approximation system

$$v \frac{\partial f(x, v, t)}{\partial x} - E(x, t) \frac{\partial f(x, v, t)}{\partial v} = 0,$$

$$\frac{\partial E(x, t)}{\partial x} = 1 - \int_{-\infty}^{\infty} f(x, v, t)\, dv. \qquad (1.9)$$

In exactly the same way, it is possible to show that if the motion of the ions is taken into account with $\varepsilon \ll \mu$, when the energy of the ions is comparable with the height of the potential barrier, close to the emitter ($x\varepsilon \ll 1$) it is possible to use the zero-th-approximation system

$$v \frac{\partial f_e}{\partial x}(x, v, t) - E(x, t) \frac{\partial f_e}{\partial v}(x, v, t) = 0,$$

$$v \frac{\partial f_i}{\partial x}(x, v, t) + \mu E(x, t) \frac{\partial f_i}{\partial v}(x, v, t) = 0, \qquad (1.10)$$

$$\frac{\partial E(x, t)}{\partial x} = \int_{-\infty}^{\infty} f_i(x, v, t)\, dv - \int_{-\infty}^{\infty} f_e(x, v, t)\, dv,$$

with the boundary conditions

$$E(x, t) = E_0(t), \quad f_e(x, v, t) = f_{0e}(v, t),$$

$$f_i(x, v, t) = f_{0i}(v, t) \quad \text{at } x=0.$$

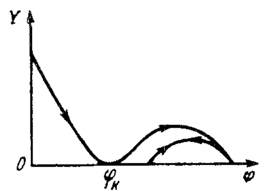

Fig. 1

Here f_i and f_e are the ion and electron distribution functions, and as density scale instead of the constant n_i we take the average density of the ions.

Fig. 2

If the energy of the ions is sufficiently large, their density n_i does not depend on the potential distribution ($n_i = n_i(t)$) and it is possible to use relations (1.8) and (1.9) if $x\varepsilon \ll 1$; in this case the scales in (1.2), including n_i, will, like n_i, be slowly varying functions of time ($\tau \ll 1/\omega_0$).

§2. System (1.10) has the integral

$$\frac{E^2}{2} = 2 \int_{\sqrt{2\varphi}}^{v_{em}} f_{0e}(v, t) \sqrt{v^2 - 2\varphi(x, t)}\, v\, dv +$$

$$+ \int_{v_{em}}^{\infty} f_{0e}(v, t) \sqrt{v^2 - 2\varphi(x, t)}\, v\, dv +$$

$$+ 2\frac{1}{\mu} \int_{\sqrt{-2\mu\varphi}}^{v_{im}} f_{0i}(v, t) \sqrt{v^2 + 2\mu\varphi(x, t)}\, v\, dv + \qquad (2.1)$$

$$+ \frac{1}{\mu} \int_{v_{im}}^{\infty} f_{0i}(v, t) \sqrt{v^2 + 2\mu\varphi(x, t)}\, v dv + C(t),$$

$$C(t) \equiv \frac{E_0^2(t)}{2} - \Big(2 \int_0^{v_{em}} f_{0e}(v, t)\, v^2\, dv + \int_{v_{em}}^{\infty} f_{0e}(v, t)\, v^2\, dv +$$

$$+ 2\frac{1}{\mu} \int_0^{v_{im}} f_{0i}(v, t)\, v^2\, dv + \frac{1}{\mu} \int_{v_{im}}^{\infty} f_{0i}(v, t)\, v^2\, dv\Big),$$

analogous to the integral (3) of system (1) in reference [8] for the stationary case. At each moment of time t_1 the potential distribution is stationary for given $E_0(t_1)$, $f_{0e}(v, t_1)$, $f_{0i}(v, t_1)$, and the time distribution $\varphi(x, t)$ is a continuous transition from one stationary potential distribution to another. This quasi-stationarity of the distribution $\varphi(x, t)$ is disturbed if for given $E_0(t)$, $f_{0e}(v, t)$, $f_{0i}(v, t)$ a stationary distribution $\varphi(x)$ does not exist or if the boundary conditions change too rapidly ($\tau \approx 1/\omega_0$). We shall ascertain what conditions can lead to disturbance of quasi-stationarity.

We write the law of conservation of emitter charge

$$\frac{dE_0}{dt} = j_{0e} - j_{0i} + j_k,$$

$$\left(j_{0e} \equiv \int_{v_{em}}^{\infty} f_{0e} v\, dv, \; j_{0i} \equiv \int_{v_{im}}^{\infty} f_{0i} v\, dv\right). \qquad (2.2)$$

Here j_k is a certain compensation current supplied to the emitter from outside. It follows from (2.2) that for quasi-stationary conditions the total current $I \equiv \equiv j_{0e} - j_{0i} + j_k$ must be very small. Otherwise rapid recharging of the emitter would occur in a time

$$\Delta t = \frac{E^\circ}{4\pi env^\circ} = \frac{mv^{\circ 2}}{er_d 4\pi env^\circ} = \frac{1}{\omega_0} \qquad \left(\Delta t \approx \frac{1}{\omega_0}\right).$$

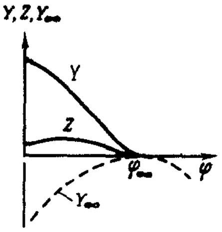

Fig. 3

Consequently, for some random small variation of the current I the electron-ion current $j_0 \equiv j_{0i} - j_{0e}$ must change so that the equality

$$j_{0e} - j_{0i} + j_k = 0 \tag{2.3}$$

is not disturbed. A change in the current j_0 takes place at the expense of a fluctuation of E_0 due to the escape of charge from the emitter, and a corresponding change in v_{em}, v_{im} at given values of $f_{0e}(v)$, $f_{0i}(v)$, since the functions $f_{0i}(v, t)$ $(v > 0)$ vary slowly in time as a condition of the problem. If $I = 0$, and E_0 varies randomly, then this produces a change in j_0, i.e., again leads to a change in I and the problem reduces to the previous one.

We shall call a regime unstable if a small change in I can produce a value of E_0 at which for given $f_{0e}(v)$, $f_{0i}(v)$ a stationary regime does not exist, and also if such a stationary regime does exist, but corresponds to a current j_0 that increases I—then any fluctuation of I, however small, can produce an automatic increase of I to finite values, which also means the disturbance of quasi-stationarity. By a stable regime we shall understand a regime in which small fluctuations of I cause changes in E_0 that affect the current j_0 in such a way that I again vanishes.

The change in I can be conveniently represented as a small change in j_k leading to a change in the charge on the emitter, and the problem consists in whether the current j_0 will then change so as to compensate the current j_k.

The absence of stationary regimes at similar values of E_0 leads to instabilities of two types. For simplicity, we shall take the case $E_0 > 0$ without slow ions. The first type of unstable regimes is shown in Fig. 1 ($Y = E^2/2$, for graphs of $Y(\varphi|x)$ see [8]). Here, since j_{0i} does not depend on the fluctuations of E_0, with decrease in j_k there should be an increase in j_{0e}, i.e., φ_{em} must decrease in order to transmit some of the slow electrons to infinity.

However, for given $f_{0i}(v)$, $f_{0e}(v)$ a decrease in φ_{em} causes a decrease in $Y(\varphi)$ over the entire interval $(0, \varphi_{em})$, i.e., in the neighborhood of φ_k Y must assume negative values, which is impossible. Thus, the regime of Fig. 1 is unstable with respect to a decrease in j_k.

The second type of unstable regimes is associated with disturbance of the condition

$$\left.\frac{dY}{d\varphi}\right|_{\varphi=\varphi_{em}} \leqslant 0 \tag{2.4}$$

necessary for Y to be positive in the neighborhood of φ_{em}.

We shall consider the function (Fig. 2)

$$Z(\varphi) \equiv \int_{v_{im}}^{\infty} f_{0i}(v)\sqrt{v^2 + 2\mu\varphi}\, v\,dv + \int_{\sqrt{2\varphi}}^{\infty} f_{0e}(v)\sqrt{v^2 - 2\varphi}\, v\,dv,$$

which at the point φ_{em} possesses the same derivative as $Y(\varphi)$; consequently, if for a given φ_{em} at specified $f_{0e}(v)$, $f_{0i}(v)$ the condition

$$\left.\frac{dZ}{d\varphi}\right|_{\varphi=\varphi_{em}} \leqslant 0 \tag{2.5}$$

is not satisfied, then a stationary regime with given φ_{em}, f_{0e}, f_{0i} is impossible. Regimes with a monotonic potential ($dY/d\varphi = 0$ at $\varphi = \varphi_{em}$, Fig. 3) are stable only if φ_{em} lies at a point of inflection of $Z(\varphi)$, when $dZ/d\varphi < 0$ close to φ_{em} (φ_1 in Fig. 2), and unstable if φ_{em} lies on an extremum of $Z(\varphi)$ or at a point of inflection with $dZ/d\varphi > 0$ close to φ_{em} (φ_2, φ_3, φ_4, φ_5 in Fig. 2).

Instability of the third type is associated with the presence of slow ions (when $E_0 > 0$) and is expressed in the fact that a small change in I corresponds to a small change in E_0 that increases the uncompensated current I and leads to its rapid growth. If we consider the relation $j_0(E_0)$ for given $f_{0e}(v)$, $f_{0i}(v)$, then an instability of the third type will occur at those E_0 for which

$$dj_0 / dE_0 < 0, \tag{2.6}$$

i.e., for example, a small additional escape of ions from the emitter due to a small increase in j_0 will produce a decrease in E_0 and hence a further increase in j_0.

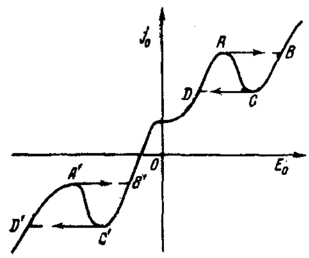

Fig. 4

We shall compute dj_0/dE_0. We have

$$j_0 = \int_{v_{im}}^{\infty} f_{0i}(v)\, v\,dv - \int_{v_{em}}^{\infty} f_{0e}(v)\, v\,dv. \tag{2.7}$$

Hence

$$\begin{gathered} dj_0 = -f_{0i}(v_{im})\,\mu\, d\varphi_{im} + f_{0e}(v_{em})\, d\varphi_{em}, \\ \varphi_{im} \equiv \mu^{-1} v_{im}^2 / 2, \qquad \varphi_{em} \equiv v_{em}^2 / 2. \end{gathered} \tag{2.8}$$

The condition $Y(\varphi_{em}) = Y(-\varphi_{im}) = 0$ for the case $E_0 > 0$ means (see 2.1)

$$0 = \int_{v_{em}}^{\infty} f_{0e}(v) \sqrt{v^2 - v_{em}^2}\, v dv + \mu^{-1} \int_0^{\infty} f_{0i}(v) \sqrt{v^2 + \mu v_{em}^2}\, v dv + \mu^{-1} \int_0^{v_{im}} f_{0i}(v) \sqrt{v^2 + \mu v_{em}^2}\, v dv + C(E_0, v_{em}, v_{im}), \quad (2.9)$$

$$0 = \int_{v_{em}}^{\infty} f_{0e}(v) \sqrt{v^2 + \mu^{-1} v_{im}^2}\, v dv + \mu^{-1} \int_{v_{im}}^{\infty} f_{0i}(v) \sqrt{v^2 - v_{im}^2}\, v\, dv + C(E_0, v_{em}, v_{im}). \quad (2.10)$$

In relations (2.9) and (2.10) the expression for $C(E_0, v_{em}, v_{im})$ has the form

$$C(E_0, v_{em}, v_{im}) = \frac{E_0^2}{2} - \Big(\int_0^{v_{em}} f_{0e} v^2 dv + \int_0^{\infty} f_{0e} v^2 dv + \mu^{-1} \int_0^{v_{im}} f_{0i} v^2 dv + \mu^{-1} \int_0^{\infty} f_{0i} v^2 dv \Big). \quad (2.11)$$

Since these equalities must be preserved when φ_{em}, φ_{im}, E_0 change, we can obtain from them the differentials

$$E_0 dE_0 = \Big[-\frac{dY}{d\varphi}(\varphi_{em}) + f_{0e}(v_{em}) v_{em} \Big] d\varphi_{em} - f_{0i}(v_{im}) \big(\sqrt{v_{im}^2 + \mu v_{em}^2} - v_{im}\big) d\varphi_{im}, \quad (2.12)$$

$$E_0 dE_0 = f_{0e}(v_{em}) \big(\sqrt{v_{em}^2 + \mu^{-1} v_{im}^2} + v_{em}\big) d\varphi_{em} + \Big[\frac{dY}{d\varphi}(\varphi_{im}) + f_{0i}(v_{im}) v_{im} \Big] d\varphi_{im}. \quad (2.13)$$

Expressing $d\varphi_{em}$, $d\varphi_{im}$ in terms of $E_0 dE_0$ and substituting the obtained expressions in (2.8), we find the relation between dj_0 and dE_0 (for the case $E_0 > 0$)

$$dj_0 = E_0 dE_0 \Big[f_{im} f_{em} v_1 (\mu + \mu^{1/2}) + \mu f_{im} Y_{em}' + f_{em} Y_{im}' \Big] \times \Big[(-Y_{em}' + f_{em} v_{em})(Y_{im}' + f_{im} v_{im}) + f_{em} f_{im} (v_{em} + v_1)(\mu^{1/2} v_1 - v_{im}) \Big]^{-1} \quad (2.14)$$

$$\frac{d\varphi_{im}}{d\varphi_{em}} = \frac{-Y_{em}' - f_{em} v_1}{Y_{im}' + \mu^{1/2} f_{im} v_1}, \qquad Y_{em}' \equiv \frac{dY}{d\varphi}(-\varphi_{em}),$$

$$Y_{im}' \equiv \frac{dY}{d\varphi}(-\varphi_{im}), \quad (2.15)$$

$$f_{em} \equiv f_{0e}(v_{em}), \qquad f_{im} \equiv f_{0i}(v_{im}),$$

$$v_1 \equiv \sqrt{v_{em}^2 + \mu^{-1} v_{im}^2}\,. \quad (2.16)$$

Similarly, for $E_0 > 0$

$$dj_0 = E_0 dE_0 \Big[-f_{im} f_{em} v_1 (\mu + \mu^{1/2}) + \mu f_{im} Y_{em}' + f_{em} Y_{im}' \Big] \times \Big[-Y_{em}' + f_{em} v_{em})(Y_{im}' + f_{im} v_{im}) + f_{em} f_{im} (\mu^{1/2} v_1 + v_{em})(v_1 - v_{em}) \Big]^{-1},$$

$$\frac{d\varphi_{em}}{d\varphi_{im}} = \frac{Y_{im}' - \mu^{1/2} f_{im} v_1}{-Y_{em}' + f_{em} v_1}. \quad (2.17)$$

Since $Y_{em}' \le 0$ and $Y_{im}' \ge 0$ for stationary regimes, we note that the denominators of (2.13)–(2.17) are nonnegative, while the numerators may be both positive and negative. For a stable regime the condition

$$dj_0 / dE_0 > 0 \quad (2.18)$$

must be satisfied.

It is easy to see that if $d\varphi_{em}/d\varphi_{im} \le 0$, then (2.18) is known to be satisfied. Condition (2.18) is not satisfied if $f_{em} = 0$, $f_{im} \ne 0$ for $E_0 > 0$ or $f_{em} \ne 0$, $f_{im} = 0$ for $E_0 < 0$ even in quite similar cases, ($f_{em} \approx 0$, $f_{im} \approx 0$). Then

$$\frac{dj_0}{dE_0} = -E_0 \frac{\mu f_{im}}{Y_{im}' + f_{im} v_{im}} \equiv -\mu \frac{E_0}{v_i^\circ}, \qquad E_0 > 0, \quad (2.19)$$

$$\frac{dj_0}{dE_0} = E_0 \frac{f_{em}}{-Y_{em}' + f_{em} v_{em}} \equiv \frac{E_0}{v_e^\circ}, \qquad E_0 < 0, \quad (2.20)$$

$$v_i^\circ = v_{im} + \frac{Y_{im}'}{f_{im}}, \qquad v_e^\circ = v_{em} - \frac{Y_{em}'}{f_{em}}.$$

A typical graph of $j_0(E_0)$ is presented in Fig. 4. The segments AC, A'C' are unstable, since between the points A, A' and the points B and B', respectively, there is a jump in the curve, and similarly between points C, C' and D, D'.

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